Revenue sharing on hierarchies

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Abstract

We consider a model of joint venture where agents are organized on a hierarchical network and each agent produces her revenue through collaborating with her superiors. We study the problem of allocating the total revenue among the agents. A hierarchy is represented by a directed tree. We investigate *superiors-reallocation-proof* allocation rules that are robust to reallocation of revenues within any coalition that includes all the superiors of its members. We obtain characterizations of superiors-reallocation-proof allocation rules imposing standard axioms in the literature of fair allocation theory.

1 Introduction

Hierarchical networks are common in real life. Hierarchies may arise groups in which agents take different responsibilities. Command or permission structures also generate hierarchies in groups. Demange [1] provides a rationale for the prevalence of hierarchies; groups obtain stability by organizing into hierarchies.

We consider a model of joint venture where agents form a hierarchical network and each agent produces her revenue through collaborating with her superiors. Individual revenue is generated not solely by an agent but only in the presence of her superiors. This can be interpreted as if there is a permission structure so that superiors supervise subordinates, or at least as

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if superiors offer advice or guidance. Our problem is to allocate the total revenue among the agents.

The allocation rules of our interest are the family of transfer rules suggested by Hougaard et al. [2] and its asymmetric variants. In transfer rules, each agent at the bottom of the hierarchy gets a fixed fraction of her revenue and transfers the rest to her immediate superiors equally. Each other agent gets the same fraction of collected revenues from her own and her subordinates, and then transfers the remainder to her immediate superiors equally. Hougaard et al. [2] also characterizes the family of transfer rules.

Hougaard et al. [2] investigate what rules are non-manipulable when a coalition of an agent and all her superiors can merge into a single representative agent or when the representative agent can split into the coalition. They show that such a merging or a splitting cannot increase the payoff under any transfer rule and moreover, these rules are the only non-manipulable rules satisfying some other standard axioms.

In this paper we consider a different type of coalitional manipulation, which enables members to freely reallocate their revenues within the coalition. Non-manipulability by such a reallocation is called reallocation-proofness. Non-manipulability is a virtue of transfer rules related to the fairness in allocation. If a rule does not have this property, some agents may seek advantageous coalitional manipulation and do not agree to get the payoff allocated by the rule. Ju [3] characterizes reallocation-proof rules in the setting where a coalition is made feasible by its connectivity in a network structure. In our investigation, a coalition can form only if they are connected, and in addition, it includes all the superiors of its members. This stricter restriction arises from the hierarchical structure and its nature of directed permission. Our non-manipulability is called superiors-reallocation-proofness. We offer new characterization results with superiors-reallocation-proofness. Some axioms in Hougaard et al. [2] are employed along with standard axioms in the literature of fair allocation theory.

Types of hierarchies we consider in this paper are extensive enough to include those represented by a directed tree. Agents do not have only one immediate superior, but may have multiple immediate superiors as long as there does not exist any undirected cycle in the hierarchy.

We characterize a family of rules called here the generalized transfer rules. They are asymmetric variants of transfer rules; two symmetric aspects of transfer rules are relaxed. First, when each agent gets a share of collected revenues from her subordinates including herself, she is allowed to take the share at her own rate, not necessarily at the same rate as other agents. Second, when each agent then transfers the rest of revenues to her immediate superiors, she is allowed to distribute it at different ratio; an immediate superior may receive more, while another immediate superior may receive less.

As in Hougaard et al. [2] , the axiomatic approach we take does not postulate a cooperative game restricted by a permission structure in hierarchy. We rather require fairness of allocation rules directly from the hierarchical structures.

2 Model

We consider a problem of allocating revenues generated collectively by a set of agents through their hierarchical collaboration. Let $N = \{1, 2, \dots, n\}$ be the set of agents. These agents form a hierarchy given by a directed tree, that is, a directed network not containing any undirected cycle. The hierarchy is represented by a correspondence $S : N \to 2^N$ that maps each agent $i \in N$ to her *immediate superiors* $S(i) \subset N$. When agent i has no immediate superior, that is, $S(i) = \emptyset$, agent i is referred to as a *top* agent. Note that there can be multiple top players. We say that player j is an *immediate subordinate* of player i if $i \in S(j)$. For agents $i, j \in N$, denote i a *superior* of j, and j a *subordinate* of i if $i = j$ or there is a finite sequence of agents $(a_1, a_2, \dots, a_m)$ such that $a_1 = i$, $a_m = j$, and $a_k \in S(a_{k+1})$ $\forall k = 1, \dots, m-1$. Let $sp : N \to 2^N$ be the correspondence that maps each agent $i \in N$ to all her superior agents (including herself). Let $sp^0(i) \equiv sp(i) \setminus \{i\}$. Similarly let $sb : N \to 2^N$ be the correspondence that maps each agent $i \in N$ to all her subordinates and $sb^0(i) \equiv sb(i) \setminus \{i\}$. Refer to a player who has no subordinate other than herself as a *bottom player*. That is, $i \in N$ is a bottom player if $sb(i) = \{i\}$. Duplet (N, S) defines a hierarchy. Throughout the paper, (N, S) is fixed.

We say that a subset of agents T is connected if for every two agents in T we can find an undirected path between them in the subset. Formally, a (undirected) *path* is a finite sequence of different agents $(a_1, a_2, \dots, a_m)$ such that either $a_k \in S(a_{k+1})$ or $a_{k+1} \in S(a_k)$ holds for all $k = 1, \dots, m-1$. Let us call this sequence a path from a_1 to a_m, and we say that agent a_k is on the path for every k. The path from a_1 to a_m is unique as the network is a directed tree. Also note that the path from a_m to a_1 is $(a_m, a_{m-1}, \dots, a_1)$.

For a subset of agents $T \subset N$, it is a (weakly) *connected* set if for every $i, j \in T$, we can find a path from i to j such that any agent on the path is a member of T.

At each position $i \in N$ of the hierarchy, agent i together with her superiors generate revenue $r_i \in \mathbb{R}_+$. Denote the profile of revenues by $r = (r_i)_{i \in N}$. The problem is to allocate the total revenue $\sum_{i \in N} r_i$ among n agents. Given a (*revenue sharing*) *problem* r, an *allocation* is a vector $x \in \mathbb{R}^n_+$ satisfying $\sum_{i \in N} x_i = \sum_{i \in N} r_i$. Call this condition *balance*. An *allocation rule* is a function $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ associating with each problem r an allocation $f(r)$.

The following notation will be used. Let $x \in \mathbb{R}^n_+$ be a vector. For all $T \subset N$, let $x(T) \equiv \sum_{i \in T} x_i$ and $x_T \equiv (x_i)_{i \in T}$. For all $i \in N$, let (x'_i, x_{-i}) be the revenue profile obtained by replacing the ith component x_i with x'_i.

We now present a family of allocation rules crucial in our investigation. Under these rules, each agent in the hierarchy keeps a fraction of her revenue and transfers the rest to her superiors. A simple case of these rules called here as *transfer rules* is suggested by Hougaard et al. [2] . To define them formally, consider first a bottom agent i. She receives a fraction of her revenue λr_i, with $\lambda \in [0, 1]$, and allocates the rest $(1 - \lambda) r_i$ equally to her immediate superiors. In the case of agent j with an immediate subordinate, the same transfer scheme applies to the total of r_j and the amount transferred to j from her immediate subordinates. If j is a top agent, she receives all her revenue r_j together with the amount transferred to her from her immediate subordinates.

This way, the transfer rule with $\lambda \in [0, 1]$ allocates revenues as follows.

If $i \in N$ is a bottom player($sb(i) = \{i\}$) ,

$$f_i(r) = \lambda r_i$$

If $j \in N$ has one superior other than herself($sb(j) \neq \{j\}$and $sp(j) \neq \{j\}$),

$$f_j(r) = \lambda \left(r_j + \sum_{l \in N:\ j \in S(l)} \frac{1}{|S(l)|} \frac{1 - \lambda}{\lambda} f_l(r) \right)$$

If $k \in N$ is a top player($sp(k) = \{k\}$),

$$f_k(r) = r_k + \sum_{l \in N:\ k \in S(l)} \frac{1}{|S(l)|} \frac{1 - \lambda}{\lambda} f_l(r)$$

More precisely, the transfer rule is defined as follows.

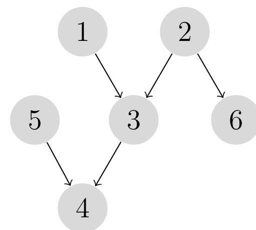

Figure 1: An example of hierarchy.

Definition. [Transfer rule]

An allocation rule is a transfer rule with λ if, for some $\lambda \in [0,1]$, it holds that for all $r \in \mathbb{R}^n_+$,

if $i \in N$ is not a top player,

$$f_i(r) = \lambda \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{l \in sp(j) \cap sb^0(i)} \frac{(1-\lambda)}{|S(l)|} \right] r_j \right],$$

and if $t \in N$ is a top player,

$$f_t(r) = r_t + \sum_{j \in sb^0(t)} \left[\prod_{l \in sp(j) \cap sb^0(t)} \frac{(1-\lambda)}{|S(l)|} \right] r_j.$$

We call the transfer rule with $\lambda = 0$ as the full transfer rule and the one with $\lambda = 1$ as the zero-transfer rule.

Example 1. Consider a set of agents $N = \{1,2,3,4,5,6\}$ with $r = (4,1,11,3,6,9)$, and $S(3) = \{1,2\}$, $S(5) = \{2\}$, $S(4) = \{3\}$, $S(6) = \{5\}$. The hierarchy is represented as in Figure 1.

The transfer rule with λ allocates $f_4(r) = 3\lambda$, $f_6(r) = 9\lambda$, $f_5(r) = 6 + \frac{3}{2}(1-\lambda)$, $f_3(r) = \lambda(11 + \frac{3}{2}(1-\lambda))$, $f_1(r) = 4 + \frac{1}{2}(1-\lambda)(11 + \frac{3}{2}(1-\lambda))$, $f_2(r) = 1 + \frac{1}{2}(1-\lambda)(11 + \frac{3}{2}(1-\lambda)) + 9(1-\lambda)$. When $\lambda = 1/3$, the allocation vector is $f(r) = (8,11,4,1,7,3)$. The full transfer rule allocates $(\frac{41}{4}, \frac{65}{4}, 0, 0, \frac{15}{2}, 0)$ and the zero-transfer rule allocates $(4,1,11,3,6,9)$.

Another family of allocation rules is a generalized version of the family of transfer rules. Generalized transfer rule allows each agent to transfer different share of collected revenues from its subordinates. Also, the amount of surplus transfered to an agent's immediate superiors may differ across them.

Definition. [Generalized Transfer rule]

An allocation rule is a generalized transfer rule with λ and μ if, for some $\lambda \equiv (\lambda_i)_{i \in N \setminus T}$ where T is a set of top players and $\lambda_i \in [0,1]$ for all i, and $\mu \equiv (\mu_i)_{i \in N \setminus T}$ where $\mu_i = (\mu_i^p)_{p \in S(i)} \in \mathbb{R}_+^{|S(i)|}$ satisfies $\mu_i^p \geq 0$ for all $p \in S(i)$ and $\sum_{p \in S(i)} \mu_i^p = 1$, it holds that for all $r \in \mathbb{R}_+^n$,

if $i \in N$ is not a top player,

$$f_i(r) = \lambda_i \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{l \in sp(j) \cap sb^0(i)} (1 - \lambda_l) \mu_l^{p(l,i)} \right] r_j \right],$$

where $p(l,i)$ is defined as the immediate superior of l that lies between l and i.

If $t \in N$ is a top player,

$$f_t(r) = r_t + \sum_{j \in sb^0(t)} \left[\prod_{l \in sp(j) \cap sb^0(t)} (1 - \lambda_l) \mu_l^{p(l,i)} \right] r_j.$$

Let us revisit the hierarchy in Example 1. The set of top players is $T = \{1,2,5\}$. When $\lambda_3 = \frac{3}{5}$, $\lambda_4 = \frac{1}{3}$, $\lambda_6 = \frac{2}{3}$ so that $\lambda = (\frac{3}{5}, \frac{1}{3}, \frac{2}{3})$, and $\mu_3^1 = \frac{3}{7}$, $\mu_3^2 = \frac{4}{7}$, $\mu_4^3 = \frac{1}{3}$, $\mu_4^5 = \frac{2}{3}$, $\mu_6^2 = 1$, so that $\mu = (\frac{3}{7}, \frac{4}{7}, \frac{1}{3}, \frac{2}{3}, 1)$, the generalized transfer rule with λ and μ allocates $f(r) = (6, \frac{20}{3}, 7, 1, \frac{22}{3}, 6)$.

Note that generalized transfer rules with λ and μ are transfer rules when for all $i \in N \setminus T$, $\lambda_i = \bar{\lambda}$ and $\mu_i^p = \frac{1}{|S(i)|}$ for all $p \in S(i)$.

3 Axioms

In this section, we introduce axioms. The first one prevents advantageous coalitional manipulation. It requires that any feasible coalition should not gain from reallocating revenues within the coalition. To be concrete, the total payoff allocated to the coalition cannot change by any reallocation of revenues among its members. In the hierarchy, an agent generates revenue by collaborating with its superiors. Superiors supervise subordinates and take responsibility of their performances. Hence, coalition is feasible only if it is connected, and contains all superiors of each member.

Let $\mathcal{F}(N,S)$ denote the set of feasible coalitions, all connected sets that contain all superiors of its members. Formally, $F \in \mathcal{F}(N,S)$ if $F \subset N$ is connected and $\forall j \in F, sup(j) \subset F$.

Definition. [Superiors-Reallocation-Proofness, SRP]

An allocation rule f satisfies *superiors-reallocation-proofness* if, for all $r \in \mathbb{R}^n_+$, all $r' \in \mathbb{R}^n_+$, and all $F \in \mathcal{F}(N, S)$, it holds that if $r(F) = r'(F)$ and $r_{N\setminus F} = r'_{N\setminus F}$, then

$$\sum_{i\in F} f_i(r) = \sum_{i\in F} f_i(r').$$

The next axiom states that any top player's revenue is irrelevant for the payoff of all other players. It is exactly the same as the one in Section 4 of Hougaard et al. [2] .

Definition. [Highest Rank Revenue Independence, HRRI] An allocation rule f satisfies *highest rank revenue independence* if, for all $r \in \mathbb{R}^n_+$, all agent $i \in N$ such that $S(i) = \emptyset$, and all $\hat{r}_i \in \mathbb{R}_+$, it holds that

$$f_{N\setminus\{i\}}(r) = f_{N\setminus\{i\}}(\hat{r}_i, r_{-i}).$$

The following property states that for each agent, only the revenues of her superiors and subordinates is relevant.

Definition. [Independence of Irrelevant Agents, IIA] An allocation rule f satisfies *independence of irrelevant agents* if, for all $i \in N$, all $r \in \mathbb{R}^n_+$, and all $\hat{r} \in \mathbb{R}_+^{|N\setminus(sp(i)\cup sb(i))|}$, it holds that

$$f_i(r) = f_i(\hat{r}, r_{sp(i)\cup sb(i)}).$$

The family of generalized transfer rules, and thus the family of transfer rules, satisfy all three axioms above.

We observe that if agent i is an immediate superior of agent j, removing the edge between them leaves us with two components.[1] N separates into two connected components, one including j and the other one including i. Pick the set of agents in the connected component including j, and denote it by $C_{j,i}$. Similarly name the subset of agents linked to i as $C_{i,j}$. Surely $C_{j,i} \cup C_{i,j} = N$. For example, see the hierarchy in Figure 2a. If we remove the edge between agent 2 and 3, two connected components consisting N are $C_{2,3} = \{2, 5, 6\}$ and $C_{3,2} = \{1, 3, 4\}$ as we observe in Figure 2b .

Consider an agent i and her immediate subordinate j. The next axiom says that if agents linked to j(members of $C_{j,i}$) transfer any surplus from

[1]Let us say a set of agents $C \subset A$ is a (connected) component of a bigger set of agents A if C is connected, and $C \cup \{i\}$ is not connected for any $i \in A \setminus C$.

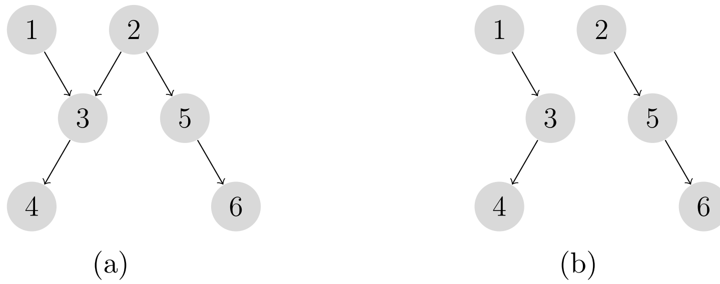

Figure 2: (a) An example of a hierarchy (N, S). (b) The edge between agent 2 and 3 is removed and N separates into two components.

the component to i, and generate zero revenue instead, then the payoffs of agents linked to i(members of $C_{i,j}$) remain unchanged. This property is very similar to that in Section 4 of Hougaard et al. [2] . The difference is that in this paper the set of agents is fixed therefore the revenue of leaving agents are nulled instead.

Definition. [Component Null-Consistency, CNC] An allocation rule f satisfies *component null-consistency* if, for all $r \in \mathbb{R}^n_+$, all $i \in N$, and all $p \in S(i)$, if r' is defined as

$$r'_j = \begin{cases} r_j + \sum_{k \in C_{i,p}} (r_k - f_k(r)), & \text{for } j = p. \\ 0, & \text{for } j \in C_{i,p}. \\ r_j, & \text{otherwise.} \end{cases}$$

then it holds that

$$f_{C_{p,i}}(r) = f_{C_{p,i}}(r').$$

The following is a standard axiom.

Definition. [Scale Invariance, SI] An allocation rule f satisties *scale invariance* if, for all $r \in \mathbb{R}^n_+$, and all $\alpha > 0$, it holds that

$$f(\alpha r) = \alpha f(r).$$

Note that the two families of generalized transfer rules and transfer rules satisfy component null-consistency and scale invariance.

We next consider two symmetry axioms. First, for a given agent i, consider the ith unit vector, $(1, 0_{-i})$. The axiom requires that every agent $i \in N$ should get the same payoff when the revenue profile is ith unit vector.

Definition. [Unit Revenue Symmetry, URS] An allocation rule f satisfies *unit revenue symmetry* if, for all $r \in \mathbb{R}^n_+$, and all $i, j \in N$, when both are not top players, it holds that

$$f_i(1, 0_{-i}) = f_j(1, 0_{-j}).$$

For the second one, consider an agent i, and her immediate superiors p and p' with identical revenues. Suppose that all agents in $C_{p,i}$ except p have zero revenue, and all agents in $C_{p',i}$ except p' have zero revenue too. Then the axiom says that the payoff of the two immediate superiors are the same.

Definition. [Superiors Symmetry, SS] An allocation rule f satisfies *superiors symmetry* if, for all $r \in \mathbb{R}^n_+$, all $i \in N$, and each pair $p, p' \in S(i)$, it holds that if $r_j = 0$ for all $j \in C_{p,i} \setminus \{p\}$ and $j \in C_{p',i} \setminus \{p'\}$, and $r_p = r_{p'}$, then

$$\sum_{l \in C_{p,i}} f_l(r) = \sum_{l \in C_{p',i}} f_l(r).$$

The two symmetries help us pin down the family of generalized transfer rules to the family of transfer rules. The last axiom states that agents with zero revenue gets zero payoff.

Definition. [No Award for Null, NA] An allocation rule f satisfies *no award for null* if, for all $r \in \mathbb{R}^n_+$, and all $i \in N$, it holds that if $r_i = 0$, then

$$f_i(r) = 0.$$

The zero-transfer rule satisfies this property whereas the family of transfer rule does not. A top player, for example, earns positive payoff if the transfer rate λ is positive and at least one of her subordinates creates positive revenue.

The followings are two examples of allocation rules that satisfy superiors-reallocation-proofness.

Example 2. Equal division rule $f_i(r) = \frac{1}{N} r(N)$ for every $i \in N$ satisfies SRP, SI, URS. It does not satisfy HRRI, IIA, CNC, SS, NA.

Example 3. Any agent who is not a top player gets $f_i = \lambda r_i$ and the top players that are superiors of i shares the rest, $(1 - \lambda) r_i$. Assume they share it equally. This rule satisfies SRP, HRRI, IIA, SI, URS, SS. It does not satisfy CNC, NA.

4 Results

We will now give a representation of rules satisfying superiors-reallocation-proofness, highest rank revenue independence, and independence of irrelevant agents. To that end, let us start with new notations. For a set of agents S, let $sb^0(S) \equiv \{i \notin S : i \in sb(j) \text{ for some } j \in S\}$. That is, an agent belongs to $sb^0(S)$ when she is not a member of S but is a subordinate of some member of S. Proposition 1 states that an allocation rule satisfies SRP, HRRI, and IIA if and only if every agent's payoff depends only on revenues of its subordinates.

Proposition 1. *Let f be an allocation rule. The following are equivalent.*

1. The rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, and Independence of Irrelevant Agents.

2. For all $F \in \mathcal{F}(N,S)$, there exists a nonnegative function $g^F : \mathbb{R}_+^{|sb^0(F)|} \to \mathbb{R}_+$ such that for all $r \in \mathbb{R}_+^n$,

$$\sum_{i \in F} f_i(r) = T^F(r(F), r_{N \setminus F}) = r(F) + g^F(r_{sb^0(F)}).$$

3. For all $i \in N$, there exists a nonnegative function $h_i : \mathbb{R}_+^{|sb(i)|} \to \mathbb{R}_+$ such that for all $r \in \mathbb{R}_+^n$,

$$f_i(r) = h_i(r_{sb(i)}).$$

Proof. ($1 \Rightarrow 2$) For a fixed subset of agents $F \in \mathcal{F}(N,S)$, let T be the set of all top players in F. Let $r \in \mathbb{R}_+^n$ be arbitrarily given. By HRRI, for every $t \in T$, $f_i(r) = f_i(0, r_{-t})$ for all $i \neq t$. By balance, we have $f_t(r) = r_t + f_t(0, r_{-t})$ for all $t \in T$. Then again by HRRI, the sum of values members of F get is

$$\sum_{i \in F} f_i(r) = \sum_{t \in T} f_t(r) + \sum_{i \in F \setminus T} f_i(r) = r(T) + \sum_{t \in T} f_t(0, r_{-t}) + \sum_{i \in F \setminus T} f_i(0_T, r_{N \setminus T}).$$

Meanwhile, by IIA, we know that $f_t(0, r_{N \setminus \{t\}}) = f_t(0_{N \setminus sb^0(t)}, r_{sb^0(t)})$ holds $\forall t \in T$. This is because for a top player, when its own revenue is fixed as zero, only the revenues of its subordinates matter. Similarly, for $i \in F \setminus T$, $f_i(0_T, r_{N \setminus T}) = f_i(0_{T \cup (N \setminus (sp(i) \cup sb(i)))}, r_{(sp(i) \setminus T) \cup sb(i)})$ holds. We note that for any member of F that is not a top player, all her superiors and subordinates are included in the collection of each member of T's subordinates. That is,

$\forall i \in F\setminus T$, $(sp(i)\setminus T)\cup sb(i) \subset \cup_{t\in T} sb^0(t)$. Then, the sum can be represented as

$$\sum_{i\in F} f_i(r) = r(T) + G^F(r_{\cup_{t\in T} sb^0(t)})$$

where G^F is a nonnegative function defined as

$$\begin{aligned} G^F(r_{\cup_{t\in T} sb^0(t)}) =& f_t(0_{N\setminus sb^0(t)}, r_{sb^0(t)}) + \\ & \sum_{i\in F\setminus T} f_i(0_{T\cup(N\setminus(sp(i)\cup sb(i)))}, r_{(sp(i)\setminus T)\cup sb(i)}). \end{aligned}$$

Now let r' be such that $r'_{F\setminus T} = 0$, $r'(T) = r(F)$, and $r'_{N\setminus F} = r_{N\setminus F}$. By SRP, $r(T) + G^F(r_{\cup_{t\in T} sb^0(t)}) = r'(T) + G^F(r'_{\cup_{t\in T} sb^0(t)})$. Then because $r(T) = r(F) - r(F\setminus T)$ and $r'(T) = r(F)$,

$$G^F(r_{\cup_{t\in T} sb^0(t)}) - r(F\setminus T) = G^F(r'_{\cup_{t\in T} sb^0(t)}).$$

We observe $\cup_{t\in T} sb^0(t) = (F\setminus T)\cup sb^0(F)$, and $sb^0(F)\cap F = \emptyset$. Then it holds that

$$G^F(r_{\cup_{t\in T} sb^0(t)}) - r(F\setminus T) = G^F(0_{F\setminus T}, r'_{sb^0(F)}) = G^F(0_{F\setminus T}, r_{sb^0(F)}).$$

Let $g^F(r_{sb^0(F)}) = G^F(0_{F\setminus T}, r_{sb^0(F)})$ and we are done.

($2 \Rightarrow 3$) Let r be arbitrarily given. If $i \in N$ is a top player, let $F = \{i\}$. By 2, $f_i(r) = r_i + g^{\{i\}}(r_{sb^0(i)})$. Define $h_i(r_{sb(i)}) = r_i + g^{\{i\}}(r_{sb^0(i)})$ and we are done.

Otherwise, assume that $i \in N$ is not a top player. Let $P \equiv \cup_{p\in S(i)} C_{p,i}$. Then $f_i(r)$ can be represented as a difference using sum over sets.

$$f_i(r) = \sum_{l\in\{i\}\cup P} f_l(r) - \sum_{p\in S(i)} \sum_{l\in C_{p,i}} f_l(r)$$

From 2 this is written as

$$f_i(r) = r_i + r(P) + g^{\{i\}\cup P}(r_{sb^0(\{i\}\cup P)}) - \sum_{p\in S(i)} [r(C_{p,i}) + g^{C_{p,i}}(r_{sb^0(C_{p,i})})],$$

which reduces to

$$f_i(r) = r_i + g^{\{i\}\cup P}(r_{sb^0(\{i\}\cup P)}) - \sum_{p\in S(i)} g^{C_{p,i}}(r_{sb^0(C_{p,i})}).$$

Observe that $sb^0(C_{p,i}) = \{i\} \cup sb^0(i) = sb(i)$ for all $p \in S(i)$, and $sb^0(\{i\} \cup P) = sb^0(i)$. Then we have

$$f_i(r) = r_i + g^{\{i\}\cup P}(r_{sb^0(i)}) - \sum_{p \in S(i)} g^{C_{p,i}}(r_{sb(i)}).$$

Because the right hand side depends only on $r_{sb(i)}$, defining $h_i(r_{sb(i)}) = r_i + g^{\{i\}\cup P}(r_{sb^0(i)}) - \sum_{p\in S(i)} g^{C_{p,i}}(r_{sb(i)})$ ends the proof.

($3 \Rightarrow 1$) HRRI and IIA are immediately verified. For SRP, let $F \in \mathcal{F}(N,S)$ and $r \in \mathbb{R}^n_+$ be arbitrarily given. In addition, let $r' \in \mathbb{R}^n_+$ be such that $r'(F) = r(F)$, and $r'_{N\setminus F} = r_{N\setminus F}$. If $i \in N \setminus F$, then i is linked to a subordinate of F.

More precisely, let us define the immediate subordinate set for F as: $K(F) = \{k \in N : k \notin F, sp(p) \subset F$ for some $p \in S(k)\}$.[2] An agent k is a member of $K(F)$ when k is not a member of F but is an immediate subordinate of a member of F. If $i \in N \setminus F$, then $i \in C_{k,q}$ for some $k \in K(F)$ and $q \in S(k) \cap F$. It follows that $sb(i) \subset N \setminus F$. From 3, we know that this implies $f_i(r) = f_i(r')$ for all $i \in N \setminus F$. Therefore, by balance $\sum_{l\in F} f_l(r) = \sum_{l\in F} f_l(r')$. □

Because a generalized transfer rule allocates each agent the value that only depends on revenues of her own and her subordinates, it is now clear that generalized transfer rules satisfy SRP, HRRI and IIA.

The following result is a characterization of the family of generalized transfer rules.

Proposition 2. *An allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, Component Null-Consistency, and Scale Invariance if and only if it is a generalized transfer rule.*

Proof. If part is easy to see. Let f be a generalized transfer rule with λ and μ. Since a generalized transfer rule allocates each agent the value that depends only on revenues of her own and her subordinates, by Proposition 1, it satisfies SRP, HRRI and IIA. SI is immediately verified. For CNC, let $i \in N$ be a player with at least one immediate superior and let $p \in S(i)$. Given $r \in \mathbb{R}^n_+$, define $x \equiv f(r)$ and let $r' \in \mathbb{R}^n_+$ be such that $r'_p = r_p + r(C_{i,p}) - x(C_{i,p})$,

[2]This K set is similar to that in the proof of theorem 1 in Demange [1].

$r'_j = 0$ for $j \in C_{i,p}$, and $r'_j = r_j$ otherwise. For a generalized transfer rule, $r(C_{i,p}) - x(C_{i,p}) = \sum_{k \in sb(i)} \left[\prod_{l \in sp(k) \cap sb(i)} (1-\lambda_l) \mu_l^{p(l,p)} \right] r_k$. Then

$$\begin{aligned}
f_p(r') &= \lambda_p \left[r'_p + \sum_{k \in sb^0(p)} \left[\prod_{l \in sp(k) \cap sb^0(p)} (1-\lambda_l) \mu_l^{p(l,p)} \right] r'_k \right] \\
&= \lambda_p \left[r_p + \sum_{k \in sb(i)} \left[\prod_{l \in sp(k) \cap sb(i)} (1-\lambda_l) \mu_l^{p(l,p)} \right] r_k \right] \\
&\quad + \lambda_p \left[\sum_{k \in sb^0(p) \setminus C_{i,p}} \left[\prod_{l \in sp(k) \cap sb^0(p)} (1-\lambda_l) \mu_l^{p(l,p)} \right] r_k \right] \\
&= \lambda_p \left[r_p + \sum_{k \in sb^0(p)} \left[\prod_{l \in sp(k) \cap sb^0(p)} (1-\lambda_l) \mu_l^{p(l,p)} \right] r_k \right] \\
&= f_p(r).
\end{aligned}$$

Therefore all agents in $C_{p,i}$ get the same value too.

For only if part, let f be an allocation rule that satisfies SRP, HRRI, IIA, CNC and SI. Let T be the set of top players. Define λ and μ as follows: $\lambda_i \equiv f_i(1, 0_{-i})$ for each $i \in N \setminus T$, and

$$\mu_i^p \equiv \begin{cases} \frac{\sum_{k \in C_{p,i}} f_k(1,0_{-i})}{1-\lambda_i}, & \text{if } \lambda_i \neq 1. \\ \frac{1}{|S(i)|}, & \text{if } \lambda_i = 1. \end{cases}$$

for each $i \in N \setminus T$ and each $p \in S(i)$. Clearly $0 \leq \lambda_i \leq 1$ and $0 \leq \mu_i^p \leq 1$ holds for such i and p. Also we observe that $\sum_{p \in S(i)} \mu_i^p = 1$ holds for each i. If $\lambda_i = 1$, there is nothing to show. If $\lambda_i \neq 1$, $\sum_{k \in C_{p,i}} f_k(1, 0_{-i}) = \mu_i^p (1-\lambda_i)$ holds.[3] By balance, $\sum_{p \in S(i)} \sum_{k \in C_{p,i} f_k(1,0_{-i})} + f_i(1, 0_{-i}) = 1$ holds because $f_j(1, 0_{-i}) = 0$ for each $j \in C_{b,i}$ for every immediate subordinate b of i. By substituting, the equality is rewritten as $(1-\lambda_i) \sum_{p \in S(i)} \mu_i^p + \lambda_i = 1$ which reduces to $\sum_{p \in S(i)} \mu_i^p = 1$.

We claim that f is a generalized transfer rule with λ and μ. Fix $r \in \mathbb{R}_+^n$. If $i \in N$ is a bottom player, by proposition 1, her payoff depends only on her revenue. Then $f_i(r) = f_i(r_i, 0_{-i}) = r_i f_i(1, 0_{-i}) = \lambda_i r_i$ by SI. For $p \in S(i)$, let $\hat{r} = (r_i, 0_{-i})$, and $x = f(r_i, 0_{-i})$. Then $\hat{r}(C_{i,p}) = r_i$. By balance and SI, $r_i = x(C_{p,i}) + x(C_{i,p}) = \mu_i^p (1-\lambda_i) r_i + x(C_{i,p})$. Therefore we have $\hat{r}(C_{i,p}) - x(C_{i,p}) = \mu_i^p (1-\lambda_i) r_i$ for every p.

Now we will show that if for an agent $i \in N$ who is neither a bottom player nor a top player, if the following two statements hold,

1. For each $j \in sb^0(i)$, $f_j(r)$ is the payoff from the generalized transfer rule with λ and μ.

[3] Note that this equality also holds even when $\lambda_i = 1$ because both sides equal zero.

2. For each $j \in sb^0(i)$, let $\tilde{r} = (r_{sb(j)}, 0_{N\setminus sb(j)})$ and $y = f(\tilde{r})$. For each $q \in S(j)$,

$$\tilde{r}(C_{j,q}) - y(C_{j,q}) = \sum_{k \in sb(j)} \left[\prod_{l \in sp(k) \cap sb(j)} (1-\lambda_l)\mu_l^{p(l,q)} \right] r_k \tag{1}$$

then the followings hold.

1. $f_i(r)$ is the payoff from the generalized transfer rule with λ and μ.
2. let $\hat{r} = (r_{sb(i)}, 0_{N\setminus sb(i)})$ and $x = f(\hat{r})$. For each $p \in S(i)$,

$$\hat{r}(C_{i,p}) - x(C_{i,p}) = \sum_{k \in sb(i)} \left[\prod_{l \in sp(k) \cap sb(i)} (1-\lambda_l)\mu_l^{p(l,p)} \right] r_k.$$

Assume that the first two statements hold, and let B be the set of immediate subordinates of i. Let $\hat{r} = (r_{sb(i)}, 0_{N\setminus sb(i)})$ and $x = f(\hat{r})$. Then

$$f_i(r) = f_i(\hat{r}) = f_i(r_i + \sum_{b \in B} (\hat{r}(C_{b,i}) - x(C_{b,i})), 0_{-i}) \tag{2}$$

holds by CNC. For $b \in B$, let $\tilde{r} = (r_{sb(b)}, 0_{N\setminus sb(b)})$ and $y = f(\tilde{r})$. Then $\hat{r}(C_{b,i}) = \tilde{r}(C_{b,i})$, and since an agent's payoff depends only on revenues of her subordinates, $x(C_{b,i}) = y(C_{b,i})$.[4] By (1) and SI, (2) equals

$$f_i(r) = \left[r_i + \sum_{b \in B} \sum_{k \in sb(b)} \left[\prod_{l \in sp(k) \cap sb(b)} (1-\lambda_l)\mu_l^{p(l,i)} \right] r_k \right] f_i(1, 0_{-i}). \tag{3}$$

Because $\lambda_i = f_i(1, 0_{-i})$, this is rewritten as

$$f_i(r) = \lambda_i \left[r_i + \sum_{k \in sb^0(i)} \left[\prod_{l \in sp(k) \cap sb^0(i)} (1-\lambda_l)\mu_l^{p(l,i)} \right] r_k \right]$$

which proves the first part.

For the second part, let us compute $\hat{r}(C_{i,p}) - x(C_{i,p})$ for every $p \in S(i)$. Obviously $\hat{r}(C_{i,p}) = (sb(i))$ and by balance, $r(sb(i)) = x(C_{p,i}) + x(C_{i,p})$ holds. By CNC and SI,

[4]If $l \in C_{b,i}$ is not a superior of b, $x_l = f_l(0_N) = 0$ because $sb(i) \cap sb(l) = \emptyset$.

$$x(C_{p,i}) = \sum_{l\in C_{p,i}} f_l(r_i + \sum_{b\in B}(\hat{r}(C_{b,i}) - x(C_{b,i}), 0_{-i})$$
$$= \left[r_i + \sum_{k\in sb^0(i)} \left[\prod_{l\in sp(k)\cap sb^0(i)}(1-\lambda_l)\mu_l^{p(l,i)}\right] r_k\right] \mu_i^p(1-\lambda_i).$$

Therefore we have $\hat{r}(C_{i,p}) - x(C_{i,p}) = \sum_{k\in sb(i)} \left[\prod_{l\in sp(k)\cap sb(i)}(1-\lambda_l)\mu_l^{p(l,p)}\right] r_k$.

By the above argument, we can sequentially show that for agents that are not top players, f gives the payoff from the generalized transfer rule with λ and μ. What remains is to verify the statement also holds for top players. Let $i \in N$ be a top player. Following the argument above similarly, we can reach the equality (3). When i is a top player, $f_i(0_{-i}, 1) = 1$ by HRRI. Then, $f_i(r) = r_i + \sum_{k\in sb^0(i)} \left[\prod_{l\in sp(k)\cap sb^0(i)}(1-\lambda_l)\mu_l^{p(l,i)}\right] r_k$ and we are done. $\square$

If we in addition add the two symmetries, the family of transfer rules is singled out from the family of generalized transfer rules.

Proposition 3. *An allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, Component Null-Consistency, Scale Invariance, Superior Symmetry, and Unit Revenue Symmetry if and only if it is a transfer rule.*

Proof. For if part, it suffices to check that a transfer rule with λ satisfies SS and URS, and both are straightforward. For only if part, let f be an allocation rule that satisfies all seven axioms. By Proposition 2, f is a generalized transfer rule. For $i \in N$ that is not a top player, since λ_i is defined as $\lambda_i \equiv f_i(0_{-i}, 1)$, by URS $\lambda_i = \lambda$ for all such i. Next, given $i \in N$, and a given pair $p, p' \in sp(i)$, suppose that r is such that $r_j = 0$ for all $j \in C_{p,i} \setminus \{p\}$ and $j \in C_{p',i} \setminus \{p'\}$, and $r_p = r_{p'}$. Because $\mu_{i,p}$ is defined as to satisfy $\sum_{l\in C_{p,i}} f_l(0_{-i}, 1) = \mu_{i,p}(1-\lambda)$, by SS, $\mu_{i,p} = \mu_{i,p'}$ should hold for all such i and pair $p, p' \in sp(i)$. Therefore $\mu_{i,p} = \frac{1}{|S(i)|}$ for all i and $p \in S(i)$. $\square$

The last result offers a characterization of zero-transfer rules. The no award for null axiom together with superiors-reallocation-proofness and highest revenue independence is strong enough to allow us drop three axioms, independence of irrelevant agents, component null consistency, and scale invariance.

Proposition 4. *An allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, and No award for Null if and only if it is the zero-transfer rule.*

Proof. Apparently, the zero-transfer rule satisfies all three axioms. For only if part, let $i \in N$ and $r \in \mathbb{R}^n_+$ be fixed. If i is a top player, we claim that $f_i(r) = r_i$. Note that by HRRI, $f_j(r) = f_j(0, r_{-i})$ for all $j \neq i$. By balance, two equalities $\sum_{l \in N} f_l(r) = r(N)$ and $\sum_{l \in N} f_l(0, r_{-i}) = r(N) - r_i$ hold. Finally, $f_i(r) = r_i + f_i(0, r_{-i}) = r_i$ holds by NA.

If i is not a top player, we claim that $f_i(r) = r_i$ holds when $f_j(r) = r_j$ holds for every $j \in sp^0(i)$. First, note that there must be at least one top player $t \in sp(i)$. Let r' be such that $r'_t = r(sp(i))$, $r_j = 0$ for $j \in sp(i) \setminus \{t\}$, and $r_{N \setminus sp(i)} = r'_{N \setminus sp(i)}$. Because $sp(i) \in \mathcal{F}(N, S)$, and f satisfies SRP, $r(sp^0(i)) + f_i(r) = r'(sp^0(i)) + f_i(r')$. By NA, then $r(sp^0(i)) + f_i(r) = r(sp(i))$ holds. Therefore, we get $f_i(r) = r_i$.

Now we are ready to show that $f_i(r) = r_i$ holds for every i that is not a top player. For given i, if all superiors of i other than herself are top players, that is, i is in the second position from the top, then $f_i(r) = r_i$. If every superior of i other than herself is either a top player or in the second position from the top, then again $f_i(r) = r_i$ holds. Repeatedly, we can show that $f_i(r) = r_i$ holds for every i wherever i lies in the hierarchy. □

5 Conclusion

In a revenue sharing model with a hierarchy, transfer rules and their asymmetric multiple-parameter extensions comprise the family of generalized transfer rules. We offer a characterization of this family with superiors-reallocation-proofness and some other axioms. As a corollary, we obtain an alternative characterization of transfer rules suggested by Hougaard et al. [2] .

Similar results establish for hierarchies where each agent has a single immediate superior. For these restricted set of hierarchies, the definition of reallocation-proofness simplifies, and independence of irrelevant agents is no longer in need for characterization.

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